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Astronomical refraction: formulas for all zenith distances

A.D. WITTMANN, Göttingen, Germany

Universitäts-Sternwarte Göttingen

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In real-time applications fast and accurate algorithms for calculating astronomical refraction are required. Some of the most widely-used expressions are fast, but numerically unstable, and can not be applied where the correction is largest, i.e. close to the horizon. In the present paper a new formula for refraction, which is both fast and numerically stable, is given and compared with previously published refraction formulas. The approximate calculation of refraction 'below the horizon', and of the finite distance correction near the horizon, are also discussed.

Key words: refraction – Earth atmosphere

AAA subject classification: 82; 36

1. Introduction

Ground-based astronomical observations at visible wavelengths are affected by refraction: When a celestial object is observed through the Earth's atmosphere, its apparent zenith distance z' will be less than its true (topocentric) zenith distance z by an amount R called 'refraction':

$$z' = z - R \tag{1}$$

R is usually small (less than 0.6°), depends on the *observed* zenith distance z', and vanishes at the zenith itself, i.e. $R(0) = 0$. In principle R is calculated as an integral over the light path through the atmosphere called the 'refraction integral': For an introduction see, e.g., Brünnow (1881), Herr und Tinter (1887), Newcomb (1906), De Ball (1912), Chauvenet (1960), Woolard and Clemence (1966), Smart (1979), or Green (1985).

The refraction integral depends on the refractive index $n = 1+x$, where x is refractivity: The latter is a function of wavelength and of the state of the atmosphere (temperature t, pressure p, etc.) at the point of integration along the light path. If Snell's law is applied to a homogeneously stratified, radially symmetric atmosphere, it can be shown that the functional dependence on altitude (i.e height above the surface) can be reduced to a dependence on the state of the atmosphere (temperature, pressure, etc.) *at the place of the observer* only, and that, to a certain approximation, R can be developed into a polynomial in powers of $\tan(z')$.

If 'zero' conditions are characterized by p_o=760 mm (1013.25 hPa) and t=0 °C (273.16 K), the dependence on atmospheric conditions is usually taken into account by multiplying the refraction R_o for 'zero' conditions by the factor

$$f_{\rm air} = \frac{(p/p_o)}{1 + \alpha t} \tag{2}$$

where $\alpha = 0.003665$ (= 1/272.85) is the volume expansion coefficient of air. This is not strictly tantamount to calculating refraction with the *actual* refractivity x (instead of x_o), and, therefore, $R = f_{\rm air} R_o$ will differ slightly from the actual refraction $R = R(x)$.

Values of the refractivity x as function of wavelength λ and relative humidity ϵ are given in Table 1, where the wavelength-dependence has been calculated from the international formula for the dispersion of dry air (Edlén 1953). The dependence on humidity is weak and is, therefore, usually ignored. In what follows I shall always assume 'visible light' (i.e. λ = 550 nm) and, for the numerical examples, ϵ = 40 %.

Table 1. Refractivity x_o (p=760.0 mm, t=0 °C) as function of wavelength λ [nm] and relative humidity ϵ [%]

λ[nm]	$\epsilon = 0$	$\epsilon = 20$	$\epsilon = 40$	$\epsilon = 60$	$\epsilon = 80$
			$t = 0°$C		
250	0.00031890	0.00031885	0.00031880	0.00031875	0.00031870
300	0.00030813	0.00030808	0.00030803	0.00030798	0.00030793
350	0.00030218	0.00030213	0.00030208	0.00030203	0.00030198
400	0.00029851	0.00029846	0.00029841	0.00029836	0.00029831
450	0.00029608	0.00029603	0.00029598	0.00029593	0.00029587
500	0.00029437	0.00029432	0.00029427	0.00029422	0.00029417
550	0.00029314	0.00029309	0.00029304	0.00029299	0.00029294
600	0.00029221	0.00029216	0.00029211	0.00029206	0.00029201
650	0.00029150	0.00029145	0.00029140	0.00029135	0.00029130
700	0.00029094	0.00029089	0.00029084	0.00029079	0.00029074
750	0.00029049	0.00029044	0.00029039	0.00029034	0.00029029
800	0.00029013	0.00029008	0.00029003	0.00028998	0.00028993
			$t = 15°$C		
250	0.00030140	0.00030127	0.00030113	0.00030100	0.00030087
300	0.00029151	0.00029137	0.00029124	0.00029111	0.00029098
350	0.00028606	0.00028593	0.00028580	0.00028567	0.00028553
400	0.00028271	0.00028257	0.00028244	0.00028231	0.00028217
450	0.00028048	0.00028034	0.00028021	0.00028008	0.00027994
500	0.00027892	0.00027878	0.00027865	0.00027852	0.00027838
550	0.00027778	0.00027764	0.00027751	0.00027738	0.00027725
600	0.00027692	0.00027679	0.00027666	0.00027652	0.00027639
650	0.00027626	0.00027613	0.00027600	0.00027586	0.00027573
700	0.00027574	0.00027561	0.00027548	0.00027534	0.00027521
750	0.00027532	0.00027519	0.00027506	0.00027493	0.00027479
800	0.00027498	0.00027485	0.00027472	0.00027458	0.00027445

Numerous theories of astronomical refraction (i.e. refraction for objects *outside* the Earth's atmosphere) have been proposed, and a great number of algorithms for its practical computation have been put forward; for an introduction see, e.g., Bruhns (1861), Wolf (1892), von Oppolzer (1901), Mahan (1962), Joshi and Mueller (1974), and Sigl (1975).

Elaborate algorithms (e.g. Bessel 1818; Garfinkel 1967) or detailed refraction tables (e.g. Bessel 1818; Abalakin 1985) may be applied if necessary. In real-time applications, however, such as, for instance, the microcomputer control of telescopes, *fast* approximation formulas are needed, be it for the sole reason of speed. Many, if not most, of the hitherto published fast approximations are based on expansions in terms of $\tan(z')$ and, therefore, are not sufficiently accurate for zenith distances of, say, $z' \geq 85°$, and can *not* be used for $z' \approx 90°$: This is not only the case for the classical expression $R = k \cdot \tan(z')$, where $k \approx 60$" is the 'refraction constant', but also for more elaborate developments in powers of $\tan(z')$ like, e.g., those of Bouguer (1729), Barfuß (1838, p. 143), Oterma (1960), Woolard and Clemence (1966, p. 83), or Saastamoinen (1972a,b). It is, therefore, the purpose of the present paper to present a newly developed semi-empirical formula which is applicable to all zenith distances, and to compare it with some of the existing 'fast' refraction formulas. In addition, an approximate formula for refraction 'below the horizon' is given, and it is shown how the refraction formulas can be used to calculate the finite distance correction of refraction: The latter must be applied to 'nearby' objects close to the horizon, i.e. where most of the tangent formulas fail.

2. Approximate refraction formulas

For the sake of simplification it is usually assumed that the atmosphere is isothermal and homogeneously stratified in concentric layers, i.e. the density ρ depends on nothing but height h in the atmosphere: Usually an exponential decrease $\rho = \rho_o \cdot \exp(-h/h_o)$ is assumed, where $h_o \approx 8$ km is the density scale height. In the present paper the Earth will be assumed spherical with radius $a = 6371$ km, and $H/a = 0.001198$ will be adopted for the ratio of the effective height of the equivalent homogeneous atmosphere to the Earth's radius, i.e. $H = 0.001198 \cdot a = 7.632$ km (which, in the present context, is not much more than a properly adjusted 'free' parameter).

2.1. Mayer's formula

Tobias Mayer (not to be confused with his son, Johann Tobias, whose first publication also dealt with refraction, cf. Mayer jr., 1781) was the first who (already in 1752) directly mapped the dependence of refraction on meteorological conditions into a refraction formula (Mayer 1770). For a detailed derivation of Mayer's formula see von Littrow (1830). With temperature converted from °*R to* °C and pressure converted from Paris inches to mm, Mayer's final formula, as quoted by Forbes (1980), reads:

$$R('') = 2.61228 \cdot p \cdot sin(z') \cdot [\sqrt{y + w^2} - w]/y^2 \tag{3}$$

Here $y = (1 + \beta \cdot t)$, $\beta = 0.00368$, $w = 16.5 \cdot \cos(z')$, and p is in mm of mercury. In the limit $z' \to 0$ Formula (3) becomes $R('') = 60.161 \cdot f_{\mathrm{air}} \cdot z'$, i.e. $x_o = 0.000291669$.

2.2. Simpson's formula

Following earlier work by Isaac Newton, Thomas Simpson (1743) assumed that the refractive index n decreases proportional to $(1 + h/a)^{-1/(m+1)}$, where h is altitude (i.e. height above the surface), and m is an empirical constant. For instance, $m = 6.46$ closely corresponds to an exponential decrease of the refractivity x with a linearly decreasing density scale height:

$$x = x_o \cdot \exp(-h/[14.1565 - 0.6049 \cdot h]) \tag{4}$$

where h is in km. This greatly simplifies the refraction integral, and it can be shown (e.g. von Littrow 1830; Barfuß1838; Mahan 1962) that to a good approximation one has:

$$R_o('') = 206264.8 \cdot [z' - \arcsin(\sin(z')/n_o{}^m)]/m \tag{5a}$$

where [...] is in radians. Modern adjusted constants for this type of formula were given in the first edition of the *Almanac for Computers* (Kaplan et al. 1976, p. B6): If customary US units of temperature (°F) and pressure (inches) are converted to °C and mm, Kaplan et al.'s formula reads

$$R_o('') = 31915.366 \cdot [z' - \arcsin(0.998115 \cdot \sin(z'))] \tag{5b}$$

In the limit $z' \to 0$ Formula (5b) becomes $R_o('') = 60.1381 \cdot z'$, i.e. $x_o = 0.000291558$.

In the 1979 edition of the *Almanac for Computers*, which was edited by L.E. Doggett (1978), a slightly revised relation has been given which, according to Trueblood and Genet (1985, p. 95), reads:

$$R_o('') = 45700.323 \cdot [z' - \arcsin(0.9986047 \cdot \sin(0.9967614 \cdot z'))] - 149.1052825 \cdot z' \tag{6}$$

where again [...] is in radians. In the limit $z' \to 0$ Formula (6) becomes $R_o('') = 62.4589 \cdot z'$, i.e. $x_o = 0.000302809$.

2.3. Saar's formula

Saar (1973) has developed a refraction formula for all zenith distances which requires the (time-consuming) evaluation of the probability integral (his eq. 10). But Saar also gave an approximation which is sufficiently accurate for $z' \leq 86°$ and numerically stable for $z' \approx 90°$. If appropriate numerical values are inserted (in particular $x_o = 0.00029$ and $h_o = 8.064$ km), Saar's formula becomes:

$$R_o('') = 2399.74 \cdot \sin(z')/\sqrt{1 + 1609.72 \cdot \cos^2(z')} \tag{7}$$

This expression is somewhat similar to the *first* formula of Tobias Mayer, which he had quoted in a letter to Leonhard Euler dated 6 January 1752.

2.4. Eisele & Shannon's formula

According to Trueblood and Genet (1985, p. 96), Eisele and Shannon (1975) have derived the following numerical fits in terms of <u>*true*</u> zenith distance z (thus requiring a few iterations to get z'):

$$R_o('') = 59.5775984 \cdot \tan(z) \cdot [1 - 1.008835086 \cdot 10^{-3} \cdot \tan^2(z)] \tag{8a}$$

for $0 \leq z \leq 85.0°$, and

$$R_o('') = 901.473336 \cdot [\exp(-0.53520501 \cdot y) + \exp(-0.107041 \cdot y)] \tag{8b}$$

for $85.0 \leq z \leq 90.6°$, where $y = (90 - z)$ is in degrees. In the limit $z' \to 0$, and with $z = z' + R$, equation (8a) becomes $R_o('') = 59.5948 \cdot z'$, i.e. $x_o = 0.000288924$.

2.5. Bennett's formula

According to Meeus (1991, p. 102), Bennett (1982) has developed the following formula:

$$R_o('') = 60.0/\tan(\pi/2 - z' + 1.351520851 \cdot 10^{-3}/[1 - 0.6069468169 \cdot z']) \qquad (9)$$

where z' is in radians. In the limit $z' \to 0$ eq. (9) will give slighly *negative* values (i.e. fairly large relative errors), and should, therefore, not be used for $z' \leq 0.5°$.

3. A new formula

The formula I shall derive in this section is semi-empirical, in the sense that it consists of an almost exact solution for a spherically symmetric, isothermal, homogeneous atmosphere of constant density (e.g. Roelofs, 1950; see Figure 1), to which is added an empirical correction for large zenith distances.

3.1. Refraction 'above the horizon'

If, in Figure 1, ψ is the angle between the normal to the atmosphere and the direction of the incoming ray at the outer border of the atmosphere (at distance $a+H$ from the centre of the Earth), ϕ is the angle between the normal to the atmosphere and the ray inside the atmosphere (i.e. the ray from the point of refraction to the observer, which makes an angle z' with the observer's zenith direction), n is the refractive index, and R is the refraction angle for an object at infinite distance (see Section 5), one has from Snell's law and the sines law in plane triangles:

$$\sin(\phi)/a = \sin(z')/(a+H) = \sin(\psi)/(a \cdot n)$$

or, with $\psi = \phi + R$:

$$n \cdot \sin(\phi) = \sin(\phi + R) = \sin(\phi) \cdot \cos(R) + \cos(\phi) \cdot \sin(R)$$

As R is always less than about 0.01 rad, we may neglect terms $\sim R^3$ and put $\sin(R) = R$ and $\cos(R) = 1 - R^2/2$. Thus we get:

$$n \cdot \sin(\phi) = \sin(\phi) - R^2/2 \ \sin(\phi) + R\cos(\phi)$$

or, solving the quadratic equation for R:

$$R = 1/\tan(\phi) \ - \ \sqrt{(1/\tan(\phi))^2 - 2 \cdot (n-1)}$$

With

$$\tan(\phi) = \sin(\phi)/\sqrt{1 - \sin^2(\phi)} = a\sin(z')/\sqrt{(a+H)^2 - [a \cdot \sin(z')]^2} = 1/\sqrt{v}$$

we get

$$R = \sqrt{v} - \sqrt{v \ - \ 2 \cdot x} \qquad (10)$$

where the abbreviation

$$v = [(1 + H/a)/\sin(z')]^2 - 1 = [1.001198/\sin(z')]^2 - 1$$

has been used, x is refractivity, and R is in radians.

Equation (10) can be applied to all zenith distances $z' > 0$, but - due to the simplified model on which it is based - is not accurate enough for large z'. The formula can, however, greatly be improved by adding an expansion in powers of v^{-2}, such that it will fit the average refraction from all other formulas mentioned above for $z' \leq 88.9°$. For $z' > 88.9°$ a linear extrapolation to the maximum possible refraction, i.e. the refraction at the *apparent* horizon (which, if $h > 0$, corresponds to $z' > 90°$), will be developed in Section 3.2. Firstly, 'above the horizon' (i.e. $z' \leq 88.9°$) the following *fast* formula for refraction is obtained:

$$R_o('') = 206264.8 \cdot [\sqrt{v} - \sqrt{v \ - \ 2 \cdot x_o}] + 1.75 \cdot 10^{-3}/v^2 \cdot (1 + 6.90 \cdot 10^{-6}/v^2) \qquad (11a)$$

for 6" $\leq z' \leq 88.9°$. The functional relationship given by eq. (11a) attains its maximum gradient $(dR/dz')_{\text{max}} = 479.2 \cdot f_{\text{air}}$ "/deg at $z' = 88.9°$; it then ceases to increase monotonically and finally reaches a value of $dR/dz' = 0$ at $z' = 90°$ (neither, of course, should it reach infinity): A different approximation should, therefore, be used for $z' > 88.9°$, which will be developed in Section 3.2. In the limit $z' \to 0$ (and with 32-bit doubleprecision accuracy) eq. (11a) remains numerically stable until z' is less than about 6"; then it is getting numerically unstable because R is computed from the difference of two almost equal, large numbers. But $z' = 6"$ is extremely small in this context, and we can, therefore, put

$$R_o('') = 206264.8 \cdot x_o \cdot z' \qquad (11b)$$

for $0 \leq z' \leq 6"$, where z' is in radians, and $R = f_{\text{air}} \cdot R_o$.

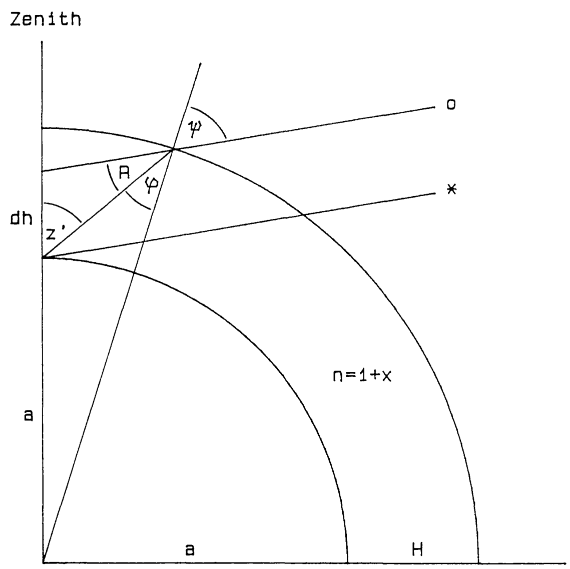

Fig. 1. Simple refraction geometry in a spherical, homogeneous atmosphere of height H and refractive index n: The incoming ray, which makes an angle ψ with the normal to the atmosphere, is deflected by the angle $R = \psi - \phi$ and is seen by the observer at zenith distance z'. Note that due to refraction a nearby object (o) having the same *true* zenith distance as an infinitely distant object (*) is actually seen slightly lower in the sky: This effect (Hansen 1838) is the same as if the observer were lifted in altitude by an amount dh (for details see text).

3.2. Refraction 'below the horizon'

If an observer is located at a certain altitude h above the surface of the Earth and has an unobstructed view to the horizon (in particular a sea horizon at h =0), he may, due to the dip of the horizon, observe celestial objects at zenith distances $z' > 90°$, i.e. below the mathematical horizon $z' = 90°$. If reasonable assumptions about the Earth's atmosphere are made it can be shown (e.g. Newcomb 1906) that the *apparent* dip, i.e. the apparent angular depression of the horizon *including* refraction, will be given by

$$dip(°) = 0.02931\sqrt{h} \tag{12}$$

where h is altitude in metres. The actual refraction in the range $z' = 90°$ to $z'_{\rm max} = 90° + dip$ is difficult to predict and may vary considerably with the state of the atmosphere; it is, however, in general *larger* than the horizontal refraction $R_{\rm hor}(h)[h > 0]$ and may reach values up to about $2 \cdot R_{\rm hor}[h = 0]$ if h is very large (e.g. Schaefer 1989): This follows from the fact that the light path 'grazes' (i.e. tangentially passes along) the surface of the Earth (h=0) at some distance d, which - with the inclusion of refraction, and for 'average' atmospheric conditions - is given by

$$d = 3.910\sqrt{h} \cdot [1 + 7.848 \cdot 10^{-8} \cdot h] \tag{13}$$

where h is in metres and d is in km. Formula (13) has been developed in accordance with occasional landmark and sunset observations made by the author at Observatorio del Teide (Canary Islands): If, for instance, $h = 2410$ m, one has $dip = 1.439°$ and $d = 192.0$ km.

If is is assumed that the maximum refraction $R_{\rm max}$ at the *apparent* horizon increases with altitude h in such a way that it fits the aformentioned observations and will asymptotically approach a value of $2R_{\rm hor}(0)$ at $z' = z'_{\rm max}$ for $h \to \infty$, and if it is further assumed that R will increase linearly from $R(90.0°)$ to $R_{\rm max}$ for $z' = 90.0° \ldots z'_{\rm max}$, we may write:

$$R = R(90°) + (z' - 90°)/dip \cdot [R_{\rm max} - R(90°)] \tag{14a}$$

where

$$R_{\rm max}('') = \frac{2 \cdot \exp(h/h_o)}{1+y} \cdot R(90°) = \frac{2 \cdot \exp(h/h_o)}{1+y} \cdot [R(88.9°) + 1.10 \cdot (dR/dz')_{\rm max}] \tag{14b}$$

and

$$y = \exp(-h/12300.0) \tag{14c}$$

Formula (14) should be applied if $90° \leq z' \leq (90° + dip)$ and $h > 0$.

For instance, if h = 2410 m and t=15 °C, one gets $R_{hor}(h)$ = 1540" and $R_{\max} = R(91.4389°)$ = 2267", so that (apart from a possible parallax) one has $z_{true} = 92.069°$ at the time of apparent crossing of the sea horizon (i.e. at the instant of rise or set). Note that Formula (14) is nothing but a reasonable approximation for 'normal' conditions, and that the actual refraction may differ widely from what is predicted. But of course applying formula (14) is better than just ignoring refraction 'below the horizon'.

Finally it might be mentioned that, for typical conditions on a seashore at sun*rise* (viz. h=0, t=10 °C), Equations (11) and (14) would predict a horizontal refraction of about 2102" and an apparent oblateness of the solar disc of 1:1.129, i.e. an apparent shrinkage of the vertical solar semidiameter (the one *above* the horizon!) by about 110".

4. Comparison of results

Table 2 shows a comparison of the results obtained from the formulas given in Sections 2 and 3 for zenith distances $z' = 5° \ldots 90°$. The calculation has been made for h=0 (p=760 mm), and for three different temperatures (t=0, t=15 and t=30 °C). R is given in seconds of arc ["], with the results from the new formula (eq. 11a) shown in the second-last column. The quantity dh (last column) will be explained in what follows (Section 5).

5. Finite distance correction

If, in Figure 1, the open circle represents the Moon (at the relatively small distance $r_{top} \approx 60$ Earth radii), and the asterisk represents a starat infinite distance from the Earth ($r_{top} = \infty$), which is observed through its *parallel* wavefronts, it is evident from Figure 1 that from the place of the observer both objects are seen at exactly the same position in the sky, whereas *without* refraction (i.e. without the Earth's atmosphere) they would be seen in different positions, with the Moon slightly *higher* in the sky than the star: This 'downward shift' of the Moon due to refraction is tantamount to placing the observer at some slightly higher altitude $h + dh$ (see Figure 1), or to a corresponding (fictitiuos!) increase of the geocentric parallax.

This correction was first noted by Hansen (1838) and is usually very small, but may become of crucial importance in eclipse and occultation calculations (or whenever differential refraction between the two bodies comes into play). It is straightforward to show (e.g. von Oppolzer, 1901, p. 551; Green, 1985, p. 90) that in case of the homogeneous, spherically symmetric atmosphere one has

$$dh = a \cdot [n \cdot \sin(z')/\sin(z' + R) - 1] \tag{15}$$

where a is the radius of the Earth, $n = 1 + x$ is the refractive index, and R is refraction. In the limit $z' \to 0$ one has $z = z' + R \to z' + (n-1) \cdot z' = n \cdot z'$, and, therefore, $dh \to 0$. Equation (15) will, however, yield correct results for small z' (and converge to $dh = 0$ with $z' \to 0$) only if R converges exactly to $x \cdot z'$: In general the latter condition will not be fulfilled because, as already stated in Section 1, the approximations $x = x(p,t)$ and $R = R_o \cdot f_{\rm air}$ are not strictly equivalent: For instance, if $R \to \alpha \cdot z'$ with $\alpha \approx 1$, one has $dh(0) \approx 1866.45 \cdot (1 - \alpha)$ [m].

This is because dh calculates as the difference of two almost equal numbers and, therefore, critically depends on the value assigned to R: The above–mentioned formulas give widely divergent results for dh if z' is 'small' (viz. $z' \leq 50°$ in the present context). A straightforward method for avoiding this numerical inconvenience is to compute (for any given refraction formula!) the quantity dh_o for a sufficiently 'small' value of z' (of the order of one or a few degrees): This will typically be of the order of a few metres only ($a \cdot [x \cdot (H/a)/(1 + x)] \approx 2.3$ m), and should then be subtracted from all values dh calculated with the *same* refraction formula. The values dh calculated from eq. (15), with R from eq. (11a), are given in the last column of Table 2.

If $r_{\rm top}$ is the topocentric distance of the body under consideration (e.g. the Moon), its apparent shift towards the horizon (in seconds of arc) will be given by

$$dz('') = 206264.8 \cdot [dh/r_{\rm top}] \cdot \sin(z) \tag{16}$$

where dh and $r_{\rm top}$ must be reckoned in the *same* units (e.g. km), and where $z = z' + R$. We then have

$$R_{\rm corrected} = R - dz \tag{17}$$

For example, at an altitude of 2410 m, a temperature of 15 °C, and for $z' = 90°$ one gets x=0.000206873, $f_{\rm air}$=0.70684, and R = 1540", so that dh = 1496 m. Taking a typical value for the *topocentric* distance of the Moon, viz. r_{top} = 383000 km, one finally gets dz = 0.81" (and a corresponding displacement of the Moon with respect to a nearby star).

Table 2. Refraction according to different authors (t is temperature in °C, x is for p=760 mm, and $\epsilon = 40$ %): [1]=Mayer, [2]=Bessel (Table), [3]=Saar, [4]=Kaplan et al., [5]=Eisele & Shannon, [6]=Doggett, [7]=Bennett, [8]=Wittmann (this work).

$z[°]$	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	$\mathrm{d}h[m]$
				$t = 0°$C $f_{\mathrm{air}} = 1.000000$ $x = 0.000293038$					
5	5.26	5.28	5.23	5.26	5.21	5.46	5.16	5.28	0
10	10.60	10.65	10.54	10.61	10.51	11.01	10.49	10.64	0
15	16.10	16.18	16.02	16.12	15.97	16.74	15.97	16.18	0
20	21.87	21.98	21.76	21.89	21.69	22.74	21.72	21.97	0
25	28.02	28.15	27.88	28.05	27.79	29.14	27.84	28.15	0
30	34.69	34.85	34.52	34.72	34.40	36.09	34.48	34.84	1
35	42.07	42.26	41.86	42.11	41.71	43.77	41.82	42.25	1
40	50.40	50.63	50.16	50.45	49.98	52.47	50.11	50.62	2
45	60.05	60.32	59.77	60.10	59.55	62.53	59.69	60.31	2
50	71.54	71.85	71.23	71.60	70.95	74.52	71.09	71.84	3
55	85.68	86.04	85.34	85.75	84.98	89.28	85.10	86.04	4
60	103.82	104.23	103.47	103.91	103.00	108.21	103.04	104.25	6
65	128.36	128.82	128.04	128.46	127.38	133.77	127.23	128.88	10
70	164.02	164.50	163.90	164.13	162.84	170.84	162.20	164.67	16
75	221.53	221.91	222.19	221.65	220.14	230.41	218.17	222.40	28
80	331.39	331.05	335.77	331.43	329.79	343.58	323.49	332.94	60
85	620.06	615.81	657.31	619.05	605.34	640.19	592.99	629.79	192
90	1985.33	2192.52	2399.74	1959.92	2215.95	2191.39	2068.65	2179.04	2220
				$t = 15°$C $f_{\mathrm{air}} = 0.947890$ $x = 0.000277512$					
5	4.98	5.00	4.96	4.99	4.94	5.18	4.89	5.01	0
10	10.04	10.07	9.99	10.05	9.96	10.44	9.94	10.09	0
15	15.26	15.30	15.19	15.28	15.14	15.87	15.14	15.33	0
20	20.73	20.79	20.63	20.75	20.56	21.56	20.59	20.83	0
25	26.55	26.63	26.43	26.59	26.34	27.62	26.39	26.68	0
30	32.87	32.96	32.72	32.91	32.61	34.21	32.69	33.03	1
35	39.86	39.97	39.68	39.91	39.54	41.49	39.64	40.05	1
40	47.76	47.88	47.55	47.82	47.37	49.73	47.50	47.98	1
45	56.90	57.05	56.66	56.97	56.45	59.28	56.58	57.17	2
50	67.79	67.95	67.52	67.87	67.25	70.64	67.38	68.10	3
55	81.19	81.37	80.89	81.28	80.55	84.63	80.67	81.56	4
60	98.37	98.57	98.08	98.49	97.62	102.57	97.67	98.82	6
65	121.61	121.81	121.37	121.77	120.73	126.80	120.60	122.17	9
70	155.37	155.51	155.36	155.58	154.33	161.93	153.75	156.08	15
75	209.79	209.68	210.62	210.10	208.62	218.40	206.80	210.81	26
80	313.57	312.44	318.27	314.16	312.46	325.67	306.63	315.59	56
85	584.75	578.68	623.05	586.80	573.00	606.83	562.09	596.97	181
90	1831.60	2079.44	2274.69	1857.79	2071.34	2077.20	1960.85	2065.49	2087
				$t = 30°$C $f_{\mathrm{air}} = 0.900941$ $x = 0.000263318$					
5	4.74	4.74	4.71	4.74	4.70	4.92	4.65	4.76	0
10	9.54	9.55	9.50	9.56	9.47	9.92	9.45	9.59	0
15	14.50	14.51	14.43	14.52	14.39	15.08	14.39	14.57	0
20	19.70	19.71	19.61	19.73	19.54	20.49	19.57	19.79	0
25	25.23	25.25	25.12	25.27	25.03	26.25	25.09	25.36	0
30	31.24	31.26	31.10	31.28	30.99	32.51	31.07	31.39	1
35	37.88	37.91	37.71	37.93	37.58	39.44	37.68	38.07	1
40	45.38	45.41	45.19	45.45	45.03	47.27	45.14	45.61	1
45	54.07	54.10	53.85	54.15	53.65	56.34	53.78	54.34	2
50	64.41	64.44	64.17	64.51	63.92	67.14	64.05	64.73	3
55	77.14	77.16	76.89	77.26	76.56	80.44	76.67	77.52	4
60	93.46	93.46	93.22	93.62	92.78	97.49	92.83	93.93	6
65	115.53	115.49	115.36	115.74	114.74	120.52	114.62	116.11	9
70	147.58	147.41	147.66	147.87	146.67	153.91	146.14	148.35	14
75	199.22	198.67	200.18	199.69	198.25	207.58	196.56	200.37	25
80	297.53	295.70	302.51	298.60	296.86	309.54	291.45	299.96	53
85	553.00	545.43	592.19	557.73	543.94	576.77	534.25	567.40	171
90	1696.74	1978.92	2162.03	1765.77	1944.97	1974.32	1863.73	1963.19	1969

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Address of the author:

Axel D. Wittmann
Universitäts-Sternwarte
Geismarlandstr. 11
D- 37083 Göttingen
Germany
e-mail: wittmann@helios.uni-sw.gwdg.deEND