☆9☆ Navigating by the Stars

B. M. Brown's complaint in the previous chapter against Cesàro's remarkable approach to spherical trigonometry might have been made by an astronomer or navigator. For the practitioner already in command of the important theorems and looking ahead to their uses in science, a pit stop to examine elegant alternative approaches is a restless, impatient exercise. While we may value the charm of beautiful mathematics on its own, its charm can only be enhanced by witnessing what it can do in some physical realization. Thus, it seems appropriate to conclude this book with an account of the life-and-death application that gave the subject much of its vitality in the past couple of centuries: finding one's position on the Earth while in a ship at sea (figure 9.1).

As far as we know, trigonometry was first used for navigation by Venetian merchant ships in the 14th century. Plying their trade through the Mediterranean and as far away as the Black Sea, Venetians used their shipping routes to establish themselves as a dominant economic power. Navigators' personal notebooks, of which several survive, recorded several navigational techniques. One of these—the table of *marteloio*—was essentially an application of plane trigonometry. How sailors managed to pick up this theory remains a mystery, although some suggest that it was altered from some of the mathematical writings of Fibonacci.

The *marteloio* is not celestial navigation; there is nothing celestial about it. It was part of a group of methods known today as "dead" (short for "deduced") reckoning, which use information about the ship's speed, direction, and time of travel to update from a previously known position to the current one. Often dead reckoning was not nearly accurate enough. During the Age of Exploration, an error of several miles easily could be the difference between a successful passage and death, either by sailing past an island containing needed provisions, or by contending with dangerous rocks off shore.

Figure 9.1. The *Flying Cloud* (1851–1874), which set the record for sailing from New York to San Francisco around Cape Horn in less than 90 days. The record stood until 1989. Drawing by Ariel Van Brummelen, based on a painting by Efren Erese.

Finding one's terrestrial latitude at sea is relatively easy: measure the altitude of the North Star above the horizon. (A more advanced and more precise technique, which uses the altitude of the Sun at noon, will be explored in the exercises.) On the other hand, the problem of determining longitude was studied already in the 16th century and would not be resolved for hundreds of years. Since longitude is measured with respect to a position chosen arbitrarily on the Earth's surface (Greenwich, England for us), any method must refer somehow to that place. Until the 18th century there was no known way to make this reference while at sea. A common navigational workaround was "parallel sailing": since one's latitude may be found via the North Star, the ship could sail along a parallel of latitude and be reasonably certain to reach the shore close to some target location.

But parallel sailing is inefficient, and where trade routes and marine power are concerned, efficiency is the key to success. So the problem of longitude remained vital to western European nations' prosperity and security. Several astronomical approaches were attempted, especially using distances measured from the center of the Moon to the Sun, a planet, or some reference star. The navigator could look up these distances in the *Nautical Almanac* (first published in 1767) as they would be seen by an observer at Greenwich, and thereby determine the time of day at Greenwich. Comparing this result with his local time gave the longitude, simply by multiplying the difference by $360°/24^h r = 15$. Navigators were lucky to have the Moon for this purpose; it was the only celestial object that moved fast enough to achieve the accuracy that was required.

However, the only person who can be said (in a sense) to have won the Longitude Prize—offered by the British government in 1714 for the first practical solution—was not a scientist, but a clockmaker. Between 1730 and 1759 John Harrison constructed a series of four chronometers capable of keeping astonishingly accurate time, even on a ship tossed by waves. Set the clock to the correct time at Greenwich; when at sea, simply use the difference between local time and Greenwich time to find the longitude. The story of Harrison's tribulations first in building the instruments, and then in convincing the government of his success (he was eventually awarded half of the money in 1765 but never officially won the prize), is so dramatic that it has been turned into a popular book and an A&E miniseries.

As successful as Harrison's timepieces were, those made by his competitors were not as reliable as his own inventions; and the best chronometers took months or even years to produce. Through the first half of the 19th century navigators usually preferred the lunar distances method. However, its use of involved mathematics taxed seamen's abilities, and nautical academies were called upon to train them in the delicate operations required to make the method work. Up to the first half of the 20th century, numerical tables were designed more and more cleverly to remove as much as possible the mathematical burden.

Preparing to Navigate: The Observations

We conclude our voyage through spherical trigonometry by exploring one of the most common techniques of determining one's position at sea, the Method of Saint Hilaire (also known as the intercept, cosine-haversine, or Davis's method), which revolutionized navigation in the late 19th century. To prepare, we must first take some observations to give us the data we need. We measure the altitude of two celestial objects above the horizon; often, but not always, one of them is the Sun. The observation usually must be made at dawn or dusk: during the day often only the Sun is visible; and at night the horizon is not visible—a bit of a hindrance when measuring altitude. Making sufficiently accurate observations on the pitching and rolling deck of a ship became possible in the 17th and early 18th centuries with improvements to sextants and quadrants. It is best to make both observations at the same time and place. Otherwise, a more complicated "running fix" procedure is required.

It is early in the evening of June 22, 2010, and we are sailing our ship eastward to the west coast of North America (figure 9.2). By dead reckoning we have a rough idea of our current position, known today as the *assumed position* or AP. In our case it is $\phi = 47°30'$ N, $\lambda = 126°45'$ W. We have encountered strong winds and may be dozens of miles away from there, but for the upcoming method to work our estimate needs to be accurate only to within about 50 nautical miles. If our AP is correct, we must travel about 100 nautical miles roughly northeast to enter the Juan de Fuca Strait between Washington state and Vancouver Island. But an error in our AP might cause us to miss the Strait's entrance altogether, so our navigational skills are required.

Figure 9.2. Our ship's assumed position. Copyright 2012 TerraMetrics, Inc. www.terrametrics.com. © 2012 Google.

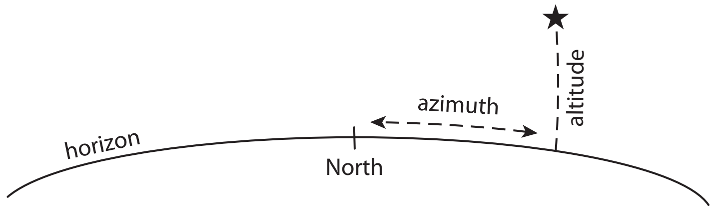

Figure 9.3. The altitude and azimuth of a star.

The sun has just set, and Venus is a bright evening star trailing the Sun in the western sky. Meanwhile, just west of south, Spica is shining brightly. So their azimuths (the direction of the object along the horizon measured from the north point; see figure 9.3) differ by about 90°. We shall see later that this is a great advantage. We check our chronometer set to Greenwich Mean Time; conveniently, it reads exactly 5:00 AM on June 23, 2010. Using our handy sextant, we measure the altitudes of our two celestial bodies; for Venus we get $h_O = 16°25.1'$ and for Spica $h_O = 28°14.1'$. We are a bit fortunate with Venus, because atmospheric refraction makes it hard to measure accurately when the object's altitude is less than 15°. Under good conditions an experienced sextant operator can measure the altitude to within 0.1 minutes of arc, so we may trust our observations to the given precision.

Now, since we are very unlikely to be exactly at the AP, our values for h_O will not quite match the altitudes at the AP; it is these differences that will allow us to fix the ship's position. So our next task is to compute the altitudes h_C of Venus and Spica at the AP, as well as their azimuths Z. In theory it is possible to observe Z directly. But in practice this can't be done accurately enough: there is no visible surface feature from which to measure either at the north point of the horizon or below the star on the horizon. Z is also an angle on the surface of the celestial sphere at the zenith, but navigational instruments measure only arcs, not angles of triangles. So we have no choice but to compute Z.

As navigators not interested in trigonometry for its own sake, we could calculate h_C and Z using nautical tables designed for this purpose. But as mathematicians, we would like to know what is going on. We appeal to the *astronomical triangle*, defined by connecting our star, the North Pole P, and the zenith Z (figure 9.4). The sides of this fundamental triangle are all familiar quantities: the complement of our known

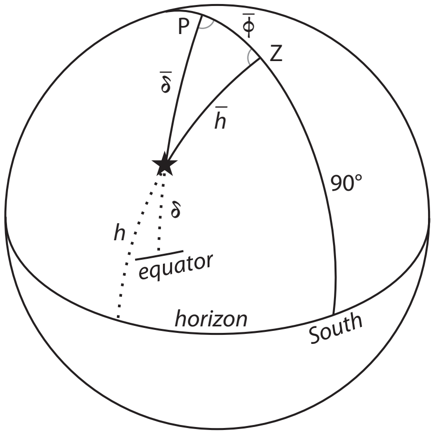

Figure 9.4. The astronomical triangle.

latitude $\bar{\varphi}$, the complement of the star's known declination $\bar{\delta}$, and the complement of the star's sought altitude $\bar{h}_c$. Two of the angles are useful as well: Z is equal to the star's azimuth, which becomes clear if we extend both of the sides departing from Z down to the horizon; and the angle at P is the star's *local hour angle t*. (The third angle, called the *parallactic angle*, will not concern us here.)

We may find the hour angle with the help of the *Nautical Almanac*, which gives us the information needed to construct an *hour angle diagram*. For Venus (as well as the Sun, Moon, and other planets), consider figure 9.5. Place point M at the top of the circle, representing the local meridian, and draw a radius connecting M to the center. Next place Greenwich G on our diagram; since our assumed longitude is $\lambda = 126°45'$ W, Greenwich's meridian is $126°45'$ *east* of ours. We turn next to the *Nautical Almanac* (see figure 9.6); it tells us that the *Greenwich hour angle* GHA of Venus at our time is $212°58.2'$. (For an online equivalent to the *Nautical Almanac*, see appendix C.) So we place Venus $212°58.2'$ counter-clockwise from Greenwich. From the diagram, then, we see that the local hour angle is $t = 212°58.2' - 126°45' = 86°13.2'$.

For Spica (or any star) the hour angle process involves an extra step. In figure 9.7, draw M and G as before. The *Nautical Almanac* tells us that the Greenwich hour angle GHA of the vernal equinox ♈, the first point of Aries, is $346°15.9'$; so we place ♈ $346°15.9'$ counter-clockwise from G. Finally, we must position the star itself on the diagram. The *Nautical Almanac* gives Spica's displacement from ♈, its *sidereal hour angle* SHA,

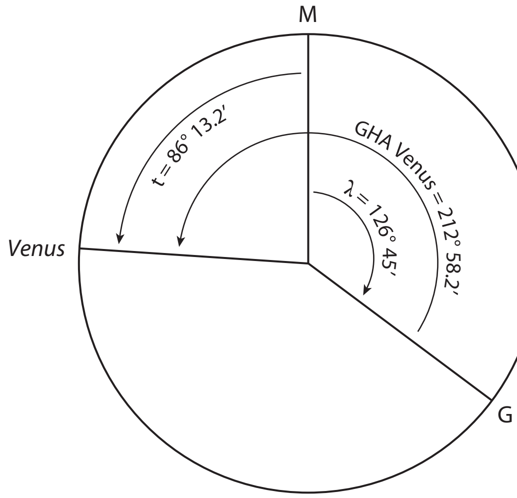

Figure 9.5. Hour angle diagram for Venus off the coast of Washington state, 5:00 a.m. GMT, June 23, 2010.

as $158°33.4'$. So, measured westward from M, Spica's local hour angle t is $-126°45' + 346°15.9' + 158°33.4' - 360° = 18°04.3'$.

A Digression: The Haversine

Now that we know three quantities in our astronomical triangle ($\bar{\delta}$, $\bar{\varphi}$, and t), solving for h_C should be a direct application of the Law of Cosines,

$$\cos \bar{h} = \cos\bar{\delta}\cos\bar{\varphi} + \sin\bar{\delta}\sin\bar{\varphi}\cos t.$$

But at sea in the early 20th century, prior to the advent of the pocket calculator, the navigator had to rely on numerical tables and hand calculation. We have seen before that logarithms were extremely useful here—they could convert the multiplication of messy trigonometric values to the much simpler task of adding them. Unfortunately, the Law of Cosines does not lend itself to logarithms. Since there is no formula for the logarithm of the sum of two quantities, the logarithm of the right side of our equation does not simplify. In practice, often the astronomical triangle was divided into two right triangles so that Napier's Rules could be applied in place of the Law of Cosines. These so-called "short methods" played well with logarithms since the Napier formulas contain

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UT	ARIES	VENUS −4.0		MARS +1.3		JUPITER −2.4		SATURN +1.1		STARS		
d h	GHA ° ′	GHA ° ′	Dec ° ′	GHA ° ′	Dec ° ′	GHA ° ′	Dec ° ′	GHA ° ′	Dec ° ′	Name	SHA ° ′	Dec ° ′
21 00	269 05.3	138 24.5	N20 14.7	109 37.7	N 9 47.5	267 00.1	S 0 26.6	89 47.4	N 2 52.2	Acamar	315 20.2	S40 15.5
01	284 07.7	153 24.0	13.9	124 38.9	47.0	282 02.3	26.5	104 49.8	52.2	Achernar	335 28.5	S57 10.7
02	299 10.2	168 23.4	13.1	139 40.1	46.4	297 04.6	26.4	119 52.2	52.1	Acrux	173 11.8	S63 09.8
03	314 12.6	183 22.9	. . 12.3	154 41.3	. . 45.9	312 06.8	. . 26.3	134 54.6	. . 52.1	Adhara	255 14.6	S28 59.3
04	329 15.1	198 22.4	11.5	169 42.5	45.3	327 09.0	26.2	149 56.9	52.0	Aldebaran	290 52.2	N16 31.8
05	344 17.6	213 21.9	10.8	184 43.7	44.8	342 11.3	26.1	164 59.3	52.0			
06	359 20.0	228 21.4	N20 10.0	199 44.8	N 9 44.2	357 13.5	S 0 26.1	180 01.7	N 2 51.9	Alioth	166 22.3	N55 54.3
07	14 22.5	243 20.9	09.2	214 46.0	43.7	12 15.7	26.0	195 04.1	51.9	Alkaid	153 00.3	N49 15.8
08	29 25.0	258 20.4	08.4	229 47.2	43.1	27 18.0	25.9	210 06.5	51.8	Al Na'ir	27 46.1	S46 54.3
M 09	44 27.4	273 19.9	. . 07.6	244 48.4	. . 42.6	42 20.2	. . 25.8	225 08.9	. . 51.8	Alnilam	275 48.9	S 1 11.7
O 10	59 29.9	288 19.4	06.9	259 49.6	42.0	57 22.4	25.7	240 11.2	51.8	Alphard	217 58.4	S 8 42.4
N 11	74 32.4	303 18.9	06.1	274 50.8	41.5	72 24.7	25.6	255 13.6	51.7			
D 12	89 34.8	318 18.4	N20 05.3	289 52.0	N 9 40.9	87 26.9	S 0 25.5	270 16.0	N 2 51.7	Alphecca	126 12.5	N26 40.8
A 13	104 37.3	333 17.8	04.5	304 53.1	40.4	102 29.1	25.4	285 18.4	51.6	Alpheratz	357 45.8	N29 08.8
Y 14	119 39.8	348 17.3	03.7	319 54.3	39.8	117 31.4	25.4	300 20.8	51.6	Altair	62 10.0	N 8 53.8
15	134 42.2	3 16.8	. . 02.9	334 55.5	. . 39.3	132 33.6	. . 25.3	315 23.2	. . 51.5	Ankaa	353 17.8	S42 14.6
16	149 44.7	18 16.3	02.1	349 56.7	38.7	147 35.8	25.2	330 25.6	51.5	Antares	112 28.6	S26 27.4
17	164 47.1	33 15.8	01.3	4 57.9	38.2	162 38.1	25.1	345 27.9	51.4			
18	179 49.6	48 15.3	N20 00.6	19 59.1	N 9 37.6	177 40.3	S 0 25.0	0 30.3	N 2 51.4	Arcturus	145 57.5	N19 07.7
19	194 52.1	63 14.8	19 59.8	35 00.3	37.1	192 42.6	24.9	15 32.7	51.4	Atria	107 31.9	S69 02.9
20	209 54.5	78 14.3	59.0	50 01.4	36.5	207 44.8	24.8	30 35.1	51.3	Avior	234 19.5	S59 32.8
21	224 57.0	93 13.8	. . 58.2	65 02.6	. . 36.0	222 47.0	. . 24.8	45 37.5	. . 51.3	Bellatrix	278 34.7	N 6 21.5
22	239 59.5	108 13.3	57.4	80 03.8	35.4	237 49.3	24.7	60 39.8	51.2	Betelgeuse	271 04.0	N 7 24.5
23	255 01.9	123 12.8	56.6	95 05.0	34.9	252 51.5	24.6	75 42.2	51.2			
22 00	270 04.4	138 12.3	N19 55.8	110 06.2	N 9 34.3	267 53.7	S 0 24.5	90 44.6	N 2 51.1	Canopus	263 57.7	S52 42.1
01	285 06.9	153 11.8	55.0	125 07.4	33.8	282 56.0	24.4	105 47.0	51.1	Capella	280 38.1	N46 00.5
02	300 09.3	168 11.3	54.2	140 08.5	33.2	297 58.2	24.3	120 49.4	51.0	Deneb	49 32.6	N45 19.0
03	315 11.8	183 10.8	. . 53.4	155 09.7	. . 32.7	313 00.5	. . 24.2	135 51.8	. . 51.0	Denebola	182 35.9	N14 30.8
04	330 14.2	198 10.4	52.6	170 10.9	32.1	328 02.7	24.2	150 54.1	51.0	Diphda	348 58.1	S17 55.5
05	345 16.7	213 09.9	51.8	185 12.1	31.5	343 04.9	24.1	165 56.5	50.9			
06	0 19.2	228 09.4	N19 51.0	200 13.3	N 9 31.0	358 07.2	S 0 24.0	180 58.9	N 2 50.9	Dubhe	193 54.3	N61 41.8
07	15 21.6	243 08.9	50.2	215 14.5	30.4	13 09.4	23.9	196 01.3	50.8	Elnath	278 15.7	N28 36.9
T 08	30 24.1	258 08.4	49.4	230 15.6	29.9	28 11.7	23.8	211 03.7	50.8	Eltanin	90 46.6	N51 29.3
U 09	45 26.6	273 07.9	. . 48.6	245 16.8	. . 29.3	43 13.9	. . 23.7	226 06.1	. . 50.7	Enif	33 49.1	N 9 55.4
E 10	60 29.0	288 07.4	47.8	260 18.0	28.8	58 16.1	23.7	241 08.4	50.7	Fomalhaut	15 26.2	S29 33.7
S 11	75 31.5	303 06.9	47.0	275 19.2	28.2	73 18.4	23.6	256 10.8	50.6			
D 12	90 34.0	318 06.4	N19 46.2	290 20.4	N 9 27.7	88 20.6	S 0 23.5	271 13.2	N 2 50.6	Gacrux	172 03.4	S57 10.7
A 13	105 36.4	333 05.9	45.4	305 21.6	27.1	103 22.9	23.4	286 15.6	50.5	Gienah	175 54.5	S17 36.2
Y 14	120 38.9	348 05.4	44.6	320 22.7	26.6	118 25.1	23.3	301 18.0	50.5	Hadar	148 50.8	S60 25.7
15	135 41.4	3 05.0	. . 43.8	335 23.9	. . 26.0	133 27.4	. . 23.2	316 20.3	. . 50.4	Hamal	328 03.4	N23 30.7
16	150 43.8	18 04.5	42.9	350 25.1	25.5	148 29.6	23.2	331 22.7	50.4	Kaus Aust.	83 46.3	S34 22.7
17	165 46.3	33 04.0	42.1	5 26.3	24.9	163 31.8	23.1	346 25.1	50.4			
18	180 48.7	48 03.5	N19 41.3	20 27.5	N 9 24.3	178 34.1	S 0 23.0	1 27.5	N 2 50.3	Kochab	137 18.9	N74 06.9
19	195 51.2	63 03.0	40.5	35 28.7	23.8	193 36.3	22.9	16 29.9	50.3	Markab	13 40.4	N15 15.7
20	210 53.7	78 02.5	39.7	50 29.8	23.2	208 38.6	22.8	31 32.2	50.2	Menkar	314 17.6	N 4 07.9
21	225 56.1	93 02.0	. . 38.9	65 31.0	. . 22.7	223 40.8	. . 22.7	46 34.6	. . 50.2	Menkent	148 10.0	S36 25.5
22	240 58.6	108 01.6	38.1	80 32.2	22.1	238 43.1	22.7	61 37.0	50.1	Miaplacidus	221 41.0	S69 45.9
23	256 01.1	123 01.1	37.3	95 33.4	21.6	253 45.3	22.6	76 39.4	50.1			
23 00	271 03.5	138 00.6	N19 36.4	110 34.6	N 9 21.0	268 47.5	S 0 22.5	91 41.7	N 2 50.0	Mirfak	308 43.9	N49 53.8
01	286 06.0	153 00.1	35.6	125 35.7	20.5	283 49.8	22.4	106 44.1	50.0	Nunki	76 00.6	S26 16.9
02	301 08.5	167 59.6	34.8	140 36.9	19.9	298 52.0	22.3	121 46.5	49.9	Peacock	53 22.0	S56 41.8
03	316 10.9	182 59.2	. . 34.0	155 38.1	. . 19.4	313 54.3	. . 22.2	136 48.9	. . 49.9	Pollux	243 30.7	N28 00.1
04	331 13.4	197 58.7	33.2	170 39.3	18.8	328 56.5	22.2	151 51.3	49.8	Procyon	245 02.3	N 5 11.8
05	346 15.9	212 58.2	32.4	185 40.5	18.2	343 58.8	22.1	166 53.6	49.8			
06	1 18.3	227 57.7	N19 31.5	200 41.7	N 9 17.7	359 01.0	S 0 22.0	181 56.0	N 2 49.8	Rasalhague	96 08.1	N12 33.2
W 07	16 20.8	242 57.3	30.7	215 42.8	17.1	14 03.3	21.9	196 58.4	49.7	Regulus	207 46.0	N11 54.9
E 08	31 23.2	257 56.8	29.9	230 44.0	16.6	29 05.5	21.8	212 00.8	49.7	Rigel	281 14.5	S 8 11.4
D 09	46 25.7	272 56.3	. . 29.1	245 45.2	. . 16.0	44 07.8	. . 21.8	227 03.2	. . 49.6	Rigil Kent.	139 54.4	S60 53.0
N 10	61 28.2	287 55.8	28.2	260 46.4	15.5	59 10.0	21.7	242 05.5	49.6	Sabik	102 14.7	S15 44.3
E 11	76 30.6	302 55.4	27.4	275 47.6	14.9	74 12.3	21.6	257 07.9	49.5			
S 12	91 33.1	317 54.9	N19 26.6	290 48.7	N 9 14.3	89 14.5	S 0 21.5	272 10.3	N 2 49.5	Schedar	349 43.2	N56 35.5
D 13	106 35.6	332 54.4	25.8	305 49.9	13.8	104 16.8	21.4	287 12.7	49.4	Shaula	96 24.4	S37 06.7
A 14	121 38.0	347 53.9	24.9	320 51.1	13.2	119 19.0	21.3	302 15.0	49.4	Sirius	258 36.0	S16 43.9
Y 15	136 40.5	2 53.5	. . 24.1	335 52.3	. . 12.7	134 21.2	. . 21.3	317 17.4	. . 49.3	Spica	158 33.4	S11 13.1
16	151 43.0	17 53.0	23.3	350 53.5	12.1	149 23.5	21.2	332 19.8	49.3	Suhail	222 54.4	S43 28.7
17	166 45.4	32 52.5	22.4	5 54.6	11.6	164 25.7	21.1	347 22.2	49.2			
18	181 47.9	47 52.1	N19 21.6	20 55.8	N 9 11.0	179 28.0	S 0 21.0	2 24.5	N 2 49.2	Vega	80 40.0	N38 47.6
19	196 50.4	62 51.6	20.8	35 57.0	10.4	194 30.2	20.9	17 26.9	49.1	Zuben'ubi	137 07.6	S16 05.2
20	211 52.8	77 51.1	19.9	50 58.2	09.9	209 32.5	20.9	32 29.3	49.1		SHA	Mer. Pass.
21	226 55.3	92 50.7	. . 19.1	65 59.4	. . 09.3	224 34.7	. . 20.8	47 31.7	. . 49.0		° ′	h m
22	241 57.7	107 50.2	18.3	81 00.5	08.8	239 37.0	20.7	62 34.0	49.0	Venus	228 07.9	14 48
23	257 00.2	122 49.7	17.4	96 01.7	08.2	254 39.2	20.6	77 36.4	48.9	Mars	200 01.8	16 38
Mer. Pass.	h m 5 58.7	v −0.5	d 0.8	v 1.2	d 0.6	v 2.2	d 0.1	v 2.4	d 0.0	Jupiter	357 49.4	6 08
										Saturn	180 40.2	17 54

Figure 9.6. A page from the *Nautical Almanac*, 2010. © British Crown copyright and/or database rights. Reproduced by permission of the Controller of Her Majesty's Stationery Office and the UK Hydrographic Office (www.ukho.gov.uk).

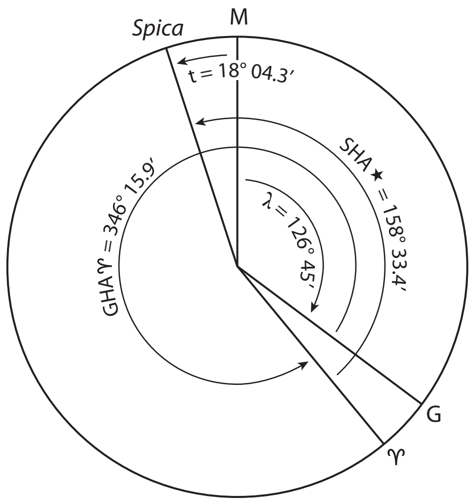

Figure 9.7. Hour angle diagram for Spica off the coast of Washington state on June 23, 2010 at 5:00 a.m. GMT.

no sums. The lack of logarithms wasn't the only problem with the Law of Cosines. If $\bar{h}$ happens to be small, then cos $\bar{h}$ changes very slowly with respect to changes in $\bar{h}$. The implication is that computing backward from cos $\bar{h}$ to $\bar{h}$ causes small rounding errors in cos $\bar{h}$ to be magnified greatly when $\bar{h}$ is found.

Necessity, the mother of invention, presses us into action. Historical navigators had more trigonometric functions available to them than we have today, and some of them have very nice properties. A few have an ancient pedigree. In addition to the sine, ancient Indian astronomers invented the "versed" (short for "reversed") sine,

$$\text{vers}\,\theta = 1 - \cos\theta.$$

Its Latin name, *sagitta* or "arrow," comes from its geometric definition (figure 9.8): if the chord of an arc is the string of a bow, the sagitta is the tip of the arrow.

One might imagine that introducing this function might simplify the trigonometry only a little, since the versed sine is just 1 minus the cosine. However, a hidden advantage comes into play with the application of a well-known identity:

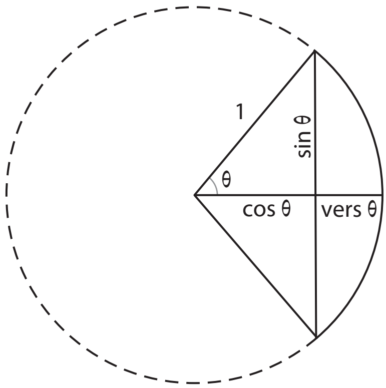

Figure 9.8. The versed sine.

$$\text{vers}\,\theta = 1 - \cos\theta = 2\sin^2\frac{\theta}{2};$$

or, altering the definition slightly by dividing by 2,

$$\text{hav}\,\theta = \tfrac{1}{2}(1-\cos\theta) = \sin^2\frac{\theta}{2}.$$

This half versed sine, or *haversine*, first tabulated by James Andrew in 1805, eventually became a favorite among seamen. A natural advantage of the haversine is that its values, the squares of sines, are always positive. This property means that a navigator never needs to worry whether the value of the haversine is positive or negative. Even better, since the haversine rises from 0 to 1 for arguments from 0° to 180°, the function is invertible in this range. So, taking the inverse of a haversine does not cause the same problems we saw in previous chapters when taking the inverse of a sine.

Another feature of the haversine recommends itself to scientists. Astronomers often work with very small arcs, for instance between two nearby stars. Imagine using the Law of Cosines on a small triangle. A quantity something like cos(0.01°) might arise; its value is 0.999999984769. If your calculator rounds to seven decimal places, it will record the cosine as 1. When the inverse cosine is taken, it will announce that the angular separation is zero! On the other hand, the haversine of 0.01° is 7.615×10^{-9}—a very small number, but not one where the rounding of significant figures will cause a problem.

The Method of Saint Hilaire

While we ventured briefly into the world of haversines, we had left our ship somewhere off the coast of Washington state needing to compute the altitude h_C of Venus and Spica. We shall follow the method of Saint Hilaire as it was updated and used in the 20th century. A career officer in the French navy, Adolphe Laurent Anatole Marcq de Blond de Saint Hilaire was captain of the School Ship *Renomée* from 1873 to 1875 when he published the papers that led to his method. He would eventually rise to Rear Admiral, and he died in 1889 while serving as Commandant of Marines in Algeria. His method is inspired by the work of his predecessor Thomas Sumner, which we shall explore in an extended exercise at the end of this chapter. Saint Hilaire's "New Navigation" was developed in the decades following the appearance of his papers. It had become established, especially in France but soon everywhere else, by the early 20th century. If one is to judge success by popularity, the New Navigation was the best of all methods; it was the standard procedure until new technologies gradually replaced all celestial methods of navigation in the second half of the 20th century.

We have enough information to find h_C, since we know three quantities in the astronomical triangle: the local hour angle $t = 86°13.2'$, Venus's declination $\delta = +19°32.4'$ (from the *Nautical Almanac*), and at least a dead reckoning value for the local latitude, $\phi = +47°30'$. We could apply the Law of Cosines, but we shall make things easier for the navigator. With haversine tables in our possession, we can manipulate the Law of Cosines into a form amenable to their use.

→We start with

$$\cos \bar{h}_C = \cos\bar{\delta} \cos\bar{\varphi} + \sin\bar{\delta} \sin\bar{\varphi} \cos t.$$

Applying the formula $\cos\theta = 1 - 2 \text{ hav } \theta$ to $\cos \bar{h}_C$ and $\cos t$, we get the ungainly

$$1 - 2 \text{ hav } \bar{h}_C = \cos\bar{\delta} \cos\bar{\varphi} + \sin\bar{\delta} \sin\bar{\varphi} - 2\sin\bar{\delta} \sin\bar{\varphi} \text{ hav } t.$$

But $\cos\bar{\delta} \cos\bar{\varphi} + \sin\bar{\delta} \sin\bar{\varphi} = \cos(\bar{\delta} - \bar{\varphi}) = \cos(\varphi - \delta)$. If we replace this latter expression with its haversine equivalent and clean up a bit, we arrive at the *haversine formula* of navigation:

$$\text{hav}\,\bar{h}_C = \text{hav}\,(\varphi - \delta) + \cos\varphi\cos\delta\,\text{hav}\,t.\rightarrow$$

In our case, the formula gives us $h_C = 16°46.3'$ for Venus (compared to $h_O = 16°25.1'$), and $h_C = 29°06.9'$ for Spica (compared to $h_O = 28°14.1'$). Of course, the reader following along with one of those rare calculators lacking a haversine button may feel free to use the Law of Cosines instead.

Now that we know all three sides and one angle of our astronomical triangle, getting the azimuth Z is just a matter of applying the Law of Sines:

$$\frac{\sin\bar{h}}{\sin t} = \frac{\sin\bar{\delta}}{\sin Z}.$$

The ambiguity that arises from needing to evaluate an arc sine is of no importance here; we have been looking at the star, and we know in what quadrant it lies. So for Venus, from $\sin Z = 0.98214$ we deduce that $Z = 79°09.3'$ west of North; and for Spica, from $\sin Z = 0.34829$ we deduce that $Z = 20°22.9'$ west of South.

Now that Z is known, we can imagine moving forward or backward in that direction on the water's surface along the *azimuth line* (figure 9.9). As we move, only Venus's altitude (not its azimuth) will change; and if we move forward far enough, we will reach Venus—or rather, we will reach the place where Venus would land if it fell directly toward the Earth's center. This point is called Venus's *geographical position*, or GP. As we move along the azimuth line, Venus's altitude will increase if we move toward Venus, or decrease if we walk away.

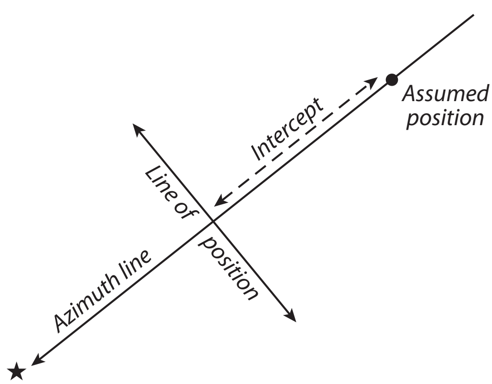

Figure 9.9. The line of position.

At some point in our journey back and forth along the azimuth line, Venus's altitude will match our observed altitude $h_O = 16°25.1'$ exactly. This point *might* be our true position. But we're not quite sure of Z, and if we turn 90° to the left or right and take a few steps, Venus will remain at the same altitude in the sky without changing Z much. In fact, we could take more than a few steps; we could travel in a giant circle centered at Venus's GP, and Venus's altitude would remain the same. (As huge as this circle is, it's not a great circle, so it's called a *small circle*.) Of course, we don't expect to need to travel very far to adjust our position, so we will assume that our true position is somewhere on the straight line perpendicular to the place on the line of azimuth where Venus's altitude matches h_O. We then draw the *line of position*, or LP, at right angles to the azimuth line, and we know that we are somewhere on that line. But how far from our AP should we travel to reach the LP?

The *intercept*, the distance from the AP to the LP, is where our method derives one of its names, and it is surprisingly easy to find. Figure 9.10 is the cross section of the universe through the center of the Earth that contains Venus. Since Venus is so far away, the lines of sight from both

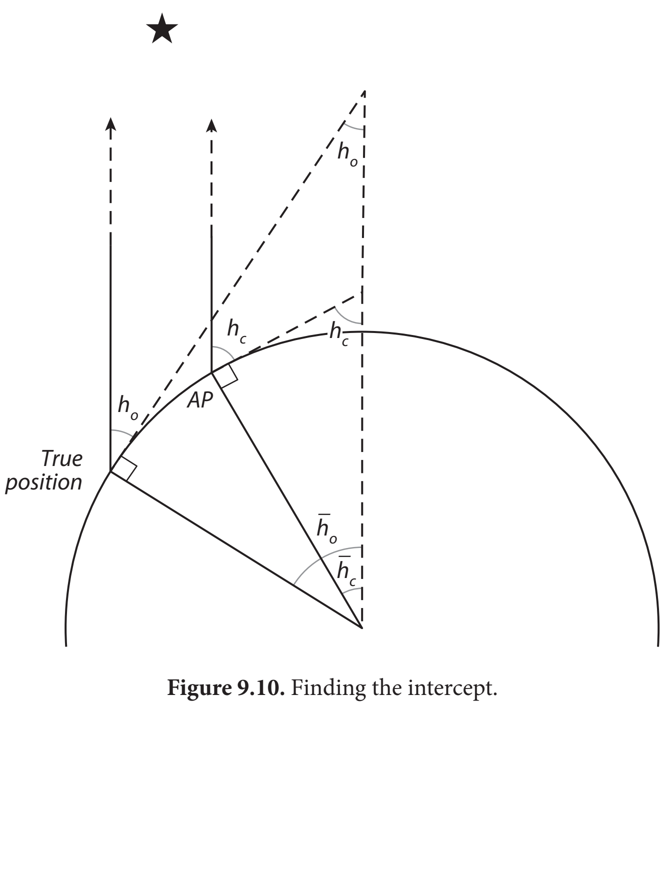

Figure 9.10. Finding the intercept.

our assumed and true positions are essentially parallel; it is the difference in position on the Earth's surface that causes h_O to differ from h_C. Form a right triangle by drawing a tangent to the circle at the AP and joining it to the line of sight from the Earth's center to Venus. The angles in this triangle will be 90°, h_C, and $\bar{h}_C$. Do the same from the true position. The angle at the center of the Earth between the assumed and true positions will be $\bar{h}_O - \bar{h}_C = h_C - h_O$. But this angle, measured in minutes of arc, is equal to the distance on the surface measured in nautical miles! So to calculate the intercept, we need only determine $60(h_C - h_O)$. In Venus's case the intercept is 21.2 nautical miles; for Spica it is 52.8 nautical miles.

We are now ready to use a plotting chart, a simple version of which is shown in figure 9.11. Our assumed position is at the center of the circle, so we may mark $\lambda = 126°45'$ W, $\phi = 47°30'$ N on the chart as in figure 9.12. Since the two vertical radii are marked off in units of 60, it is convenient to assume that the circle has a radius of 60 nautical miles (if the intercepts had been smaller, we could have used a smaller scale). So

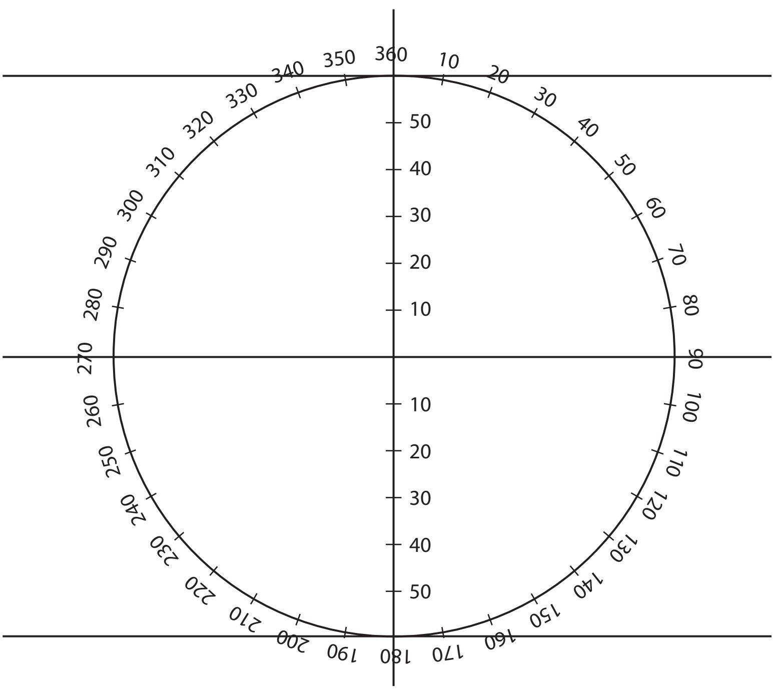

Figure 9.11. A simple blank nautical chart.

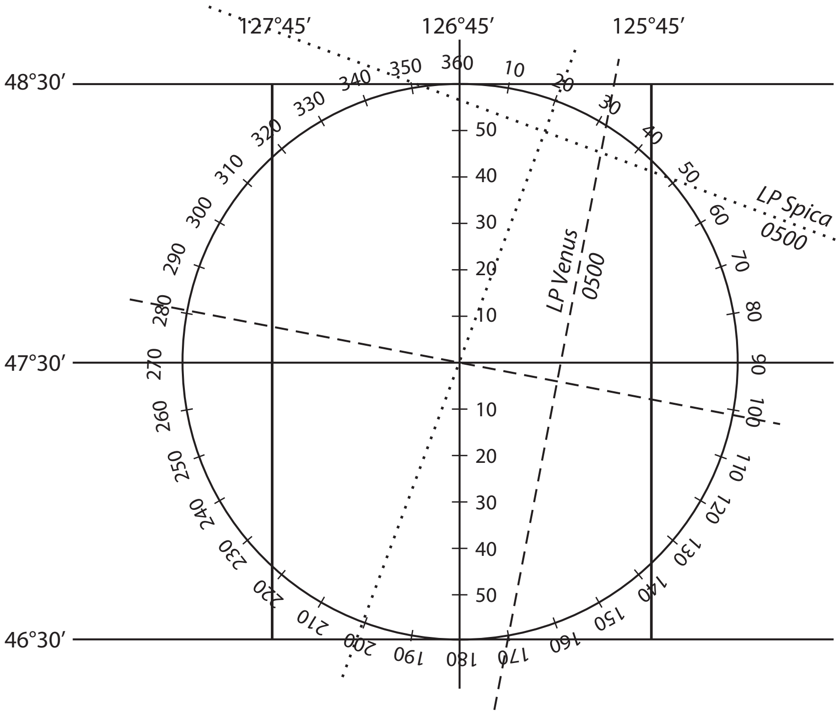

Figure 9.12. A nautical chart containing the fix for our ship.

we mark the top and bottom vertical lines 60 nautical miles above and below 47°30′, at $\phi = 48°30'$ and $\phi = 46°30'$. The longitude scale, however, is different. From exercise 9 of chapter 2, recall that the east-west distance corresponding to one degree of longitude decreases as one moves north, according to the cosine of the latitude. We can work out this scale cleverly without a needing a calculator to compute the cosine: mark two places on the circle 47°30′ up and down from the rightmost point of the circle, and draw a vertical line. Do the same on the left. The three vertical lines will each be 1° apart in longitude.

Earlier we calculated Venus's azimuth to be 79°09.3′ west of North, so we draw the azimuth line onto our chart. The intercept is 21.2 nautical miles, so we must move that distance away from the center of the circle. But in which direction? In this case we must travel away from (rather than toward) Venus or disaster will ensue. As seen on figure 9.10, if $h_C > h_O$ then we must move away from Venus, and if $h_C < h_O$ we must move toward it. Navigators remember this rule by memorizing the phrase "computed greater away." Now that we have located Venus's intercept (to the right of and a little below the center), we draw a perpendicular. This marks Venus's line of position (LP), and we know that our ship is somewhere along it.

Of course, one LP isn't enough to pin down our location, but we had the foresight to make two observations. So we leave the reader to repeat the process for Spica and get a second LP. The intersection of the two LP's is our *fix*, our best estimate of our true position. Occasionally navigators make three observations and draw three LP's. Since the three LP's are unlikely to intersect at precisely the same point but instead form a small triangle, the navigator assumes that the ship is located at the most dangerous point within the triangle. Better safe than sorry.

Figure 9.12 shows our resulting fix. We can now see why taking observations of two objects with azimuths differing by about 90° was such a good idea: our LP's are almost at right angles to each other, producing a much more precise intersection point than if the LP's had been nearly parallel. In our chart, we find that our ship is actually around 55 nautical miles northeast of the AP, at about $\phi = 48°15'$N, $\lambda = 126°00'$W; this position is indicated in figure 9.13. We are much closer to Juan de Fuca Strait than we thought (less than 50 nautical miles rather than 100), and we need to approach the Strait heading almost due east, rather than northeast. It's a good thing we have a navigator on board.

Figure 9.13. The assumed position of our ship, and the true position northeast of it. Copyright 2012 TerraMetrics, Inc. www.terrametrics.com. © 2012 Google.

Now the secret may be revealed. The true position in this example, from which the altitude observations were obtained using astronomical software, is exactly $\phi = 48°15'$ N, $\lambda = 126°00'$ W. The true position is so close to our fix that the thickness of the lines at the intersection of the two LP's covers both locations. We have pinpointed our ship to a distance of less than 1000 feet.

Exercises

1. Finding one's terrestrial latitude is as easy as measuring the altitude of the North Star, but sailors often used a more accurate method called the "noon sight." Near local noon in the northern hemisphere, the Sun crosses the meridian (the great circle through the north and south points of the horizon and the zenith) in the south, reaching its maximum altitude. For a number of minutes around noon its altitude is almost constant. The sailor repeatedly measures the Sun's altitude near noon, and considers the *noon sight* to be the largest measured value.
   (a) Use the concepts from chapter 2 to explain how this measurement determines the local latitude. One quantity from the *Nautical Almanac* is needed; which one?
   (b) On June 23, 2011, a sailor gets a noon solar altitude of $60°25.1'$. What is the local latitude? (Use the *Nautical Almanac*, paper or online, to get the quantity you need.)
2. Make an hour angle diagram for Mars and Altair using your local longitude, for June 22, 2010 at 0900 GMT. Use the page from the *Nautical Almanac* reproduced in figure 9.6.
3. (a) Since the haversine formula is an alternate formulation of the Law of Cosines, it clearly applies to any triangle, not just the astronomical one. Express the formula in terms of a general triangle with sides a, b, c and angles A, B, C.
   (b) Solve $a = 52°$, $b = 39°$, $c = 44°$ using the haversine formula.
4. (a) Show that $\sin a \sin b = \text{hav}(a+b) - \text{hav}(a-b)$. (*Hint:* Use the cosine addition and subtraction formulas.)
   (b) Substitute this result into the equation you generated in question 3(a), to obtain the following formula that involves *only* haversines:

$$\text{hav}\, c = \text{hav}(a-b) + [\text{hav}(a+b) - \text{hav}(a-b)]\,\text{hav}\, C.$$

   [Nielsen/Vanlonkhuyzen 1944, 119]

5. The formula derived in the previous exercise may be used to build a device called the *haversine nomogram*, capable of solving some spherical triangles visually. Make a scale as in figure E-9.5.1, where the position of each tick mark corresponds to the haversine of that angle. (The more tick marks you can make, the more accurate your result.) Align three of these scales in a rectangle opened at the top, as in figure E-9.5.2. Imagine that the triangle has sides $a = 87°$, $b = 52°$, and $c = 106°$. Then $a - b = 35°$ and $a + b = 139°$. Draw a diagonal line from 35° on the left scale to 139° on the right scale. Then draw a horizontal line from the 106° point on the right scale and move down to the bottom scale when you reach the diagonal line. The angle at that place, 115°, is the value of C.
(a) Solve the triangle of question 3(b) using a haversine nomogram.
(b) Explain why this method produces the correct answer. (*Hint:* use the formula of question 4(b), solved for hav C.)
(c) Devise a method to use a haversine nomogram to find the third side if two sides and their included angle are given. [Nielsen/Vanlonkhuyzen 1944, 120–121]

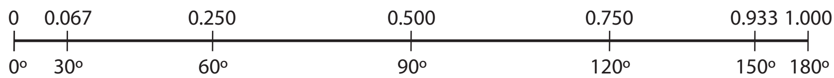

Figure E-9.5.1. The haversine nomogram.

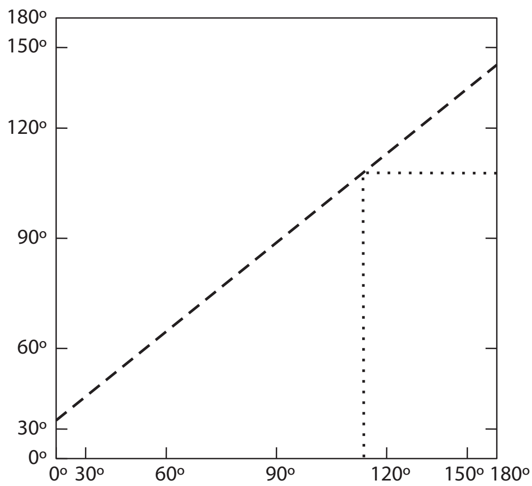

Figure E-9.5.2. Finding an angle in a triangle with three known sides using a haversine nomogram.

6. It is early evening on June 22, 2010 and you are somewhere southeast of the coast of Long Island, NY, hoping to sail toward Rhode Island. Your chronometer reads June 23, 2010, 1:00 AM GMT, and your assumed position is $\phi = 40°05'$, $\lambda = 70°33'$. A little west of south you spot Antares, and with your sextant you measure it to be $16°34.0'$ above the horizon. Just north of west is Venus, with an altitude of $18°40.1'$. Use Saint Hilaire's method to determine your position. (Figure 9.6 contains the appropriate page from the *Nautical Almanac*. The solution is $\phi = 40°25'$, $\lambda = 71°14'$.)
7. Make up your own navigation problem. Do this with astronomical software as follows: choose true and assumed positions with values of φ and λ less than one degree apart. In your software, set your location to the *true* position, find a time near sunrise or sunset when two objects are visible with azimuths separated by around 90°, record their altitudes, and note the time in GMT. Now discard the true position, and proceed with Saint Hilaire's method. You may use the online *Nautical Almanac* if necessary. When you are finished, compare your fix with the true position.
8. Perform the Saint Hilaire calculations in this chapter, but use the Law of Cosines directly on the astronomical triangle rather than the haversine formula. Round all trigonometric quantities to three decimal places for both methods. Assuming that you have a haversine button on your calculator, which method is faster? Does one give a more accurate result than the other?
9. (Assumes calculus) Find the derivative of the Sun's altitude with respect to local hour angle. Explain from the result why solar observations taken when the Sun is in the East or West were preferred to when the Sun is in the South (near noon). [courtesy of Joel Silverberg]
10. *Sumner's method:* In the late morning of December 17, 1837 Thomas Hubbard Sumner was approaching St. George's Channel between Ireland and Wales on his way to Scotland, having departed three weeks earlier from South Carolina. Unsure of his position since his last fix 600 miles back and dealing with bad weather conditions, he was fearful of encountering the dangerous rocks on the southeast tip of Ireland. The critical checkpoint that Sumner needed to locate was Small's Light just off the coast of Wales; if he could sail toward it, he would be able to find safe passage through the channel (figure E-9.10). Suddenly the clouds parted momentarily and gave him a brief opportunity to measure the Sun's altitude. Spurred by necessity, he had a flash of insight that led to his new method of navigation, and eventually inspired Saint Hilaire's method as

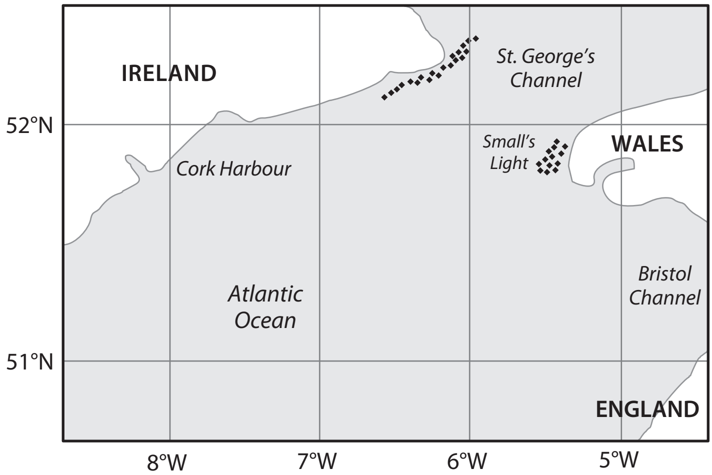

Figure E-9.10. Sumner's Method.

well. In this exercise we shall reproduce his discovery as he described it in 1843.

(a) Sumner's fundamental formula on the astronomical triangle is equivalent to the Law of Cosines, but it is in a form that makes logarithmic calculation easier:

$$\text{vers}\, t = 1 - \cos t = \{\cos(\phi - \delta) - \sin h\} \sec\phi \sec\delta.$$

Explain why this formula is easier to use with logarithms, and derive it from the Law of Cosines.

(b) By dead reckoning Sumner believed his latitude to be somewhere around $\phi = 51°37'$ N. Decrease this to 51°. From the *Almanac* we know the Sun's declination to be $\delta = -23°23'$. At the moment when the clouds parted, Sumner observed the Sun's altitude to be $h = 12°10'$. Use this data and the formula in (a) to determine the hour angle t. You do not need to use logarithms.

(c) In time units, you should have found that $t = 1^h 43^m 59^s$, which represents the time before local noon. However, Sumner needed to account for the *equation of time*, a small effect that accounts for the fact that the Sun does not quite travel through the celestial sphere at a constant speed. On the date of Sumner's observation the equation of time was $3^m 37^s$, which implied that the apparent time had to be adjusted $3^m 37^s$ earlier. Sumner's chronometer told him that the time was 10:47:13 AM in Greenwich.

What is the difference between local time and Greenwich time? Multiply by 15 to get the ship's longitude. Plot the resulting ship's position on the map and call it point *A*.

(d) The above calculations are based on a latitude of 51°, which is 37′ less than Sumner's best estimate. Repeat the calculations of (b) and (c), this time for a latitude of 52°. Plot the new position as point *B*.

(e) Draw a line through *A* and *B*. Drawn correctly, the line should pass through or very close to Small's Light. Since the Sun's altitude is the same at both *A* and *B*, it will also be the same at every point on the line joining *A* and *B*. (To be precise, *A* and *B* both lie on the *line of position*, a very large—but not great—circle containing all the points on the Earth's surface where the Sun's altitude is 12°10′.) In what direction is the azimuth of the Sun with respect to this line?

Sumner reasoned correctly that whatever his true latitude was, he had to be somewhere on the line of position. Since (luckily) the line passes through Small's Light, Sumner simply sailed in the direction of his line. He soon encountered Small's Light, passed safely through St. George's Channel, and changed the history of navigation. [thanks to Joel Silverberg]

Where to Go from Here

Our tour through the world of spherical trigonometry has ended, but there are countless journeys that may be taken from here. Todhunter and Leathem's 1907 textbook and Casey's 1889 treatise are particularly rich sources for further exploration of mathematical topics:

- the properties of small circles (not necessarily small in stature, but not great circles) on their own, or inscribed in and circumscribed around spherical triangles;
- a duality between theorems on small circles and on great circles;
- Hart's Circle, a spherical analog to the nine point circle in plane geometry;
- approximate formulas and the use of calculus to determine variations in quantities when certain other quantities are varied (useful in geodesy and other practical applications).

Also in the nineteenth century, spherical trigonometry became subsumed into a more general trigonometry that included non-Euclidean spaces. Although this did not affect the classroom and we have chosen to skip over it here, the interested reader will find the theory both powerful and fascinating. Seth Braver's *Lobachevski Illuminated* is an extensively annotated translation of one of the earliest works in this area.

The reader may wish to explore extensions of spherical trigonometry in astronomy and navigation; in the literature of those subjects you will find many variants to the procedures shown here and even entirely new approaches. In astronomy, consider W. M. Smart's *Textbook on Spherical Astronomy* or Simon Newcomb's *Compendium of Spherical Astronomy*; in navigation, consult Charles Cotter's *History of Nautical Astronomy.* If you care to linger a while in these dusty old textbooks, you will find that the playground of spherical trigonometry contains many more forgotten delights.